Toposes

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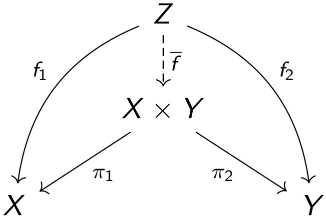
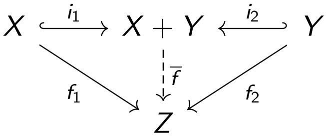
Overview

1. Building up to toposes
   - Limits and colimits
   - Exponenetiation
   - Subobject classifiers
2. Toposes and their logic
   - Toposes
   - Heyting algebras

Products



Coproducts



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- $e : E \to X$ is an equaliser if for any commuting diagram

$$Z \xrightarrow{h} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

there is a unique $\overline{h}$ such that triangle commutes:

$$\begin{array}{ccccc} E & \xrightarrow{e} & X & \underset{g}{\overset{f}{\rightrightarrows}} & Y \\ {\scriptstyle \overline{h}}\uparrow & \nearrow_{h} & & & \\ Z & & & & \end{array}$$

Equalisers - an example

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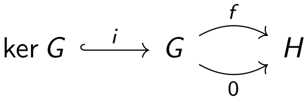

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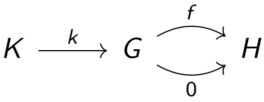

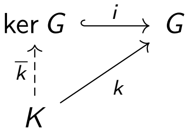

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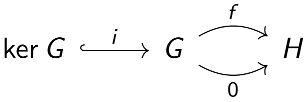

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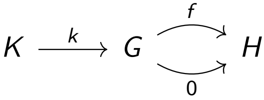

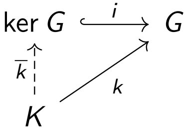

- $\overline{k} : K \to \ker(G)$, $\overline{k}(g) = k(g)$.

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- $q : Y \to Q$ is a coequaliser if for any commuting diagram

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{h} Z$$

there is a unique $\overline{h}$ such that the triangle commutes:

$$\begin{array}{ccccc} X & \underset{g}{\overset{f}{\rightrightarrows}} & Y & \xrightarrow{q} & Q \\ & & & \searrow^{h} & \downarrow{\scriptstyle \overline{h}} \\ & & & & Z \end{array}$$

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- Let $\mathbb{Z}$ be a group under addition. The following is a coequaliser diagram:

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- $\overline{h}(g) = h(g)$. Look familiar?

Pullbacks

- Take functions $f : X \to Z$, $g : Y \to Z$. The pullback of f and g the following set:

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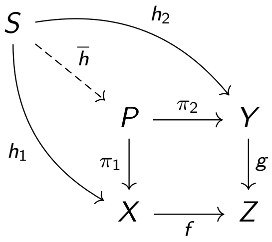
- As a diagram:

$$\begin{array}{ccc} P & \xrightarrow{\pi_2} & Y \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

Pullbacks

- The universal property:



Pullbacks - Intersection

- Take two subsets $i : 2\mathbb{Z} \hookrightarrow \mathbb{Z}$ and $j : 3\mathbb{Z} \hookrightarrow \mathbb{Z}$. Their pullback is their intersection:

$$\begin{array}{ccc} 6\mathbb{Z} & \overset{i^*}{\hookrightarrow} & 3\mathbb{Z} \\ {\scriptstyle j^*}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ 2\mathbb{Z} & \underset{i}{\hookrightarrow} & \mathbb{Z} \end{array}$$

Pushouts

- Given two functions with same domain $f : X \to Y$, $g : X \to Z$, we can from their pushout square:

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ {\scriptstyle g}\downarrow & & \\ Y & & \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{f} & Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle q_2} \\ Y & \xrightarrow[q_1]{} & Q \end{array}$$

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- Q is $(Y + Z)/\sim$, where $\sim$ is generated by $\sim = \langle\{(f(x), g(x)) : x \in X\}\rangle$.

Pushouts

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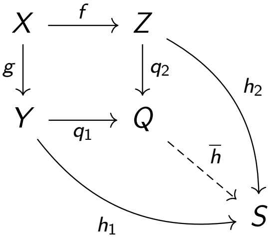

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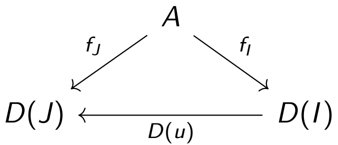
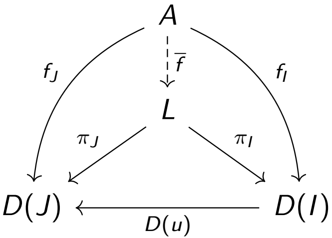
- A cone on D is an object $A \in \mathscr{A}$ with maps $(A \xrightarrow{f_I} D(I))_{I \in \mathbf{I}}$, such that the following commutes for all $u : I \to J$ in $\mathbf{I}$.



Limits

- A limit of D is a universal cone $(L \xrightarrow{\pi_I} D(I))_{I \in \mathbf{I}}$ such that the following commutes for all $I, J \in \mathbf{I}$:



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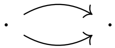

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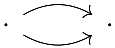

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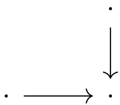

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- **FinSet** has finite limits and finite colimits.

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- $\mathscr{A}$ has exponentials if $\mathscr{A}$ has exponentials for all $X \in \mathscr{A}$.

Universal property of exponentials

- A category having exponentials is equivalent to the following universal property:

$$
\begin{array}{ccccc}
Y & \qquad & Y \times X & & \\
\Big\downarrow{\scriptstyle \overline{f}} & & \Big\downarrow{\scriptstyle \overline{f} \times 1_X} & \searrow^{f} & \\
Z^X & & Z^X \times X & \xrightarrow[\epsilon]{} & Z
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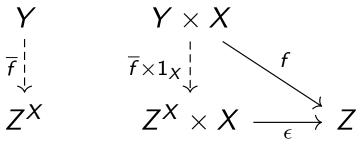

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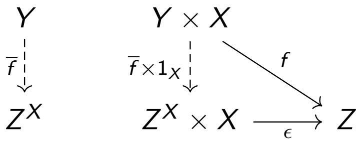

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- In **Set** it is an evaluation map.

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- We can embed $\mathbb{R}$ into $\mathbb{R}^2$ (as vector spaces) with the injective linear map $f(x) = (x, 0)$.
- The map $g(x) = (2x, 0)$ is 'the same' embedding.

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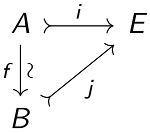

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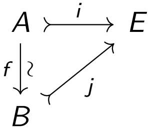

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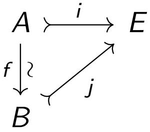

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- What about subobjects of topological spaces?

Characteristic maps

- A subset $i : U \hookrightarrow X$ corresponds with map $\chi_i : X \to 2$, where

$$\chi_i(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{otherwise.} \end{cases}$$

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- $\mathcal{P}(X) \cong \mathrm{Sub}(X) \cong \mathbf{Set}(X, 2)$.
- This isomorphism lets us classify subobjects with maps into 2.

Subobject classifiers

- Let $\mathscr{A}$ be a category with terminal object 1. A subobject classifier is a map $\mathrm{True} : 1 \to \Omega$ such that for any subobject $m : U \rightarrowtail X$, there exists a unique map $\chi_m : X \to \Omega$ such that the following is a pullback diagram:

$$\begin{array}{ccc} U & \overset{m}{\rightarrowtail} & X \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \chi_m} \\ 1 & \underset{\mathrm{True}}{\longrightarrow} & \Omega \end{array}$$

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- A sieve on $A \in \mathscr{A}$ is a set of arrows with codomain A, that is closed under right composition.
- Let $\Omega : \mathscr{A}^{op} \to \mathbf{Set}$ be the functor sending A to the set of sieves on A.

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- For a fixed group G, True : $1 \to 2$ is a subobject classifier for $G -$ **Set**. (2 has trivial group action)
- For a small category $\mathscr{A}$, $[\mathscr{A}^{op}, \mathbf{Set}]$ has a subobject classifier.
- A sieve on $A \in \mathscr{A}$ is a set of arrows with codomain A, that is closed under right composition.
- Let $\Omega : \mathscr{A}^{op} \to \mathbf{Set}$ be the functor sending A to the set of sieves on A.
- $(\text{True}_A : 1(A) \to \Omega(A))_{A \in \mathscr{A}}$ is a natural transformation that selects the maximal sieve on A.

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- $G -$ **Set** is a special case of $[\mathscr{A}^{op}, \mathbf{Set}]$, where $\mathscr{A}$ is G regarded as a category.

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- For any topos $\mathscr{E}$ and A in $\mathscr{E}$, $\mathscr{E}/A$ is a topos.

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- Their logic is intuitionistic - proof by contradiction isn't always valid.

The End