A Survey of Infinity Toposes

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Abstract

This paper provides a brief overview of certain ideas and definitions in ∞-topos theory. In particular, we look at the definitions of both Grothendieck and elementary ∞-toposes and look at some properties they have.

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1 Introduction

The category of sets is the archetypal topos and arguably the archetypal category. It provides a framework that most of classical mathematics can be viewed in and has a sufficiently strong internal logic to for most mathematical purposes. This logic, however, is extensional in the sense that "$x = y$" is either true or false and contains no additional structure. Upgrading our logic to a dependent type theory can endow the identity type withe structure of a higher groupoid. Ordinary 1-category theory isn't sufficient to capture this notion, so instead we turn to higher category theory. Here, the category of sets is replaced with a different category - the infinity category of spaces, which becomes the archetypal ∞-topos. It is conjectured that dependent type theories, such as homotopy type theory, are the internal languages of such categories, which motivates their study from a logical perspective.

There are many different ways to describe what an ∞-topos is. If we take the view that a 1-topos is a 'place where you can do mathematics', then an ∞-topos can be described as a 'place where you can do homotopy theory'. These slogans can be made slightly more precise by saying that an ∞-topos is to the ∞-category of spaces what a 1-topos is to the category of sets. We will look at a common definition of the ∞-category of spaces and define presheaves and sheaves. We then will look at object classifiers and finish by discussing some properties of elementary ∞-toposes.

Throughout this paper, an ∞-category will be an $(\infty, 1)$-category, with the same abuse of notation for toposes.

2 ∞-toposes

To get started with ∞-toposes, we need to define the ∞-category of spaces.

Definition 2.1. The ∞-category $\mathcal{S}paces$ is defined to be $N_\Delta(\mathcal{K}\mathrm{an})$ and is called the **∞-category of spaces**. The simplicial category $\mathcal{K}\mathrm{an}$ is the full subcategory of **sSet** spanned by Kan complexes.

For this definition to make sense, we require $\mathcal{S}paces$ to be a ∞-category, which can be shown using the following lemmas.

Lemma 2.2. *The simplicial category $\mathcal{K}an$ is locally Kan.*

Proof. Letting X, Y be objects in $\mathcal{K}\mathrm{an}$, we have a simplicial set $\mathcal{K}\mathrm{an}(X, Y) = Y^X$, which by Proposition 2.3 is a simplicial set. It can also be shown that this is a Kan complex. Then, by a theorem of Cordier and Porter (lecture - 12/11/19), $N_\Delta(\mathcal{K}\mathrm{an})$ is a ∞-category. □

Proposition 2.3. *[Lur09, Proposition 1.2.7.3.1] Let X be a simplicial set and Y a ∞-category. Then the simplicial set of Y^X is a ∞-category.*

Proof. We see that by using the cartesian closed structure of **sSet**, the left lifting problem has a solution if and only if the right lifting problem has a solution:

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & Y^X \\ \big\downarrow & & \\ \Delta^n & & \end{array} \qquad\qquad \begin{array}{ccc} X \times \Lambda^n_k & \longrightarrow & Y \\ \big\downarrow & & \\ X \times \Delta^n & & \end{array}$$

The map $X \times \Lambda^n_k \to X \times \Delta^n$ is formed from an inner anodyne map $\Lambda^n_k \to \Delta^n$ and a monic $1_X : 1_X$. By [Lur09, Corollary 2.2.5.4], the map $X \times \Lambda^n_k \to X \times \Delta^n$ is an inner anodyne map. As Y is a ∞-category, the right diagram has a lift, so Y^X is a ∞-category. $\square$

The higher category $\mathcal{S}paces$ plays the role that the category of sets does in ordinary category theory. We use it to define higher presheaves.

Definition 2.4. Let $\mathcal{C}$ be a small ∞-category. We define $\widehat{\mathcal{C}}$ to be the ∞-category $\mathcal{S}paces^{\mathcal{C}^{op}}$, also known as the **$\infty$-category of ∞-presheaves on $\mathcal{C}$**.

When the context is clear, we will just refer to ∞-presheaves just as presheaves.

Recalling from 1-category theory, a Grothendieck topos is a category equivalent to the category of sheaves on a site: $\mathrm{Sh}(\mathbb{A}, \tau)$. This category comes with an inclusion map that has a left exact, left adjoint. It turns out that this gives an equivalent definition of Grothendieck toposes [Rez10, Proposition 3.5], which is the one we shall generalise for higher toposes.

Definition 2.5. [Lur09, Defintion 6.1.0.4] Let X be an ∞-category. We say X is an **∞-topos** if there exists a small ∞ category $\mathcal{C}$ and an accessible left exact localization functor $\widehat{\mathcal{C}} \to X$.

We shall briefly look at the definition of each adjective in this definition.

2.1 Accessibility

Accessibility requires a lot of prior work and definitions to define, so we will instead discuss some ideas and consequences that it brings. The idea behind accessibility is a way to handle certain categories which are too big or not essentially small. We will present a definition for completeness, but we will say no more about the condition.

Definition 2.6. [nLa19a, Definition 2.1.2] An ∞-category X is **κ-accessible** if the following hold:

1. X is locally small,
2. X has κ-filtered colimits,
3. The full subcategory $X \hookrightarrow X$ of κ compact objects (3.5) is an essentially small ∞-category.
4. $X \hookrightarrow X$ generates X under κ-filtered colimits.

2.2 Localization

Recall what it means for a functor of infinity categories to be fully faithful:

Definition 2.7. [Lur09, Definition 1.2.10.1] A map $f : X \to Y$ of infinity categories is **fully faithful** when the induced map $X^r(x, y) \to Y^r(f(x), f(y))$ is a weak homotopy equivalence for all $x, y \in X_0$.

Definition 2.8. [Lur09, Definition 5.2.7.2] A functor $f : X \to Y$ between infinity categories is a **localization** if f has a fully faithful right adjoint.

Unsurprisingly, this is the same definition that we have for 1-categories, with the exception that 'fully faithful' be interpreted correctly. This can be thought of as a higher characterisation of reflective subcategories.

2.3 Left exact

We see that the definition of left exact is lifted from the 1-categorical setting without much tweaking.

Definition 2.9. [Lur09, Remark 5.3.2.10] Let X, Y be ∞-categories and suppose X has all finite limits. A functor $F : X \to Y$ is **left exact** if it preserves finite limits.

As we see, Definition 2.5 can simplified to the following: X is an ∞-topos if X is accessible and we have an adjunction

$$\widehat{\mathcal{C}} \underset{i}{\overset{a}{\underset{\perp}{\rightleftarrows}}} X$$

for some small ∞-category $\mathcal{C}$, with i an embedding, and a commuting with finite limits.

3 Object classifiers

In 1-category theory, toposes can be shown/defined to have a subobject classifier, which classifies the set of monics in the topos. This means that there is an object Ω which represents the presheaf Sub. We will generalise this notion so we can talk about object classifiers, which ∞-toposes can be shown to have.

We note that in 1-category theory, from a category $\mathscr{C}$ we can define a category $\mathscr{C}_{Mono}$ whose objects are monomorphisms and whose maps are pullback squares with horizontal maps being monic. It is clear that $\mathscr{C}$ has a subobject classifier if and only if $\mathscr{C}_{Mono}$ has a terminal object. This is the property that we will abstract to define object classifiers.

Definition 3.1. [Lur09, Notation 6.1.3.4] Let X be an ∞-category and S a subclass of morphisms which is stable under pullback. We define the following (∞) categories:

1. O_X^S is the full subcategory of $X^{(\Delta^1)}$ spanned by S.
2. $O_X^{(S)}$ is the subcategory of O_X whose objects are elements of S and whose morphisms $f \to g$ are pullback diagrams where f is the pullback of g along some map in X.

Definition 3.2. [Lur09, Definition 6.1.6.1] Let X be an infinity category with pullbacks and S a collection of morphisms of X which is stable under pullback. A morphism f **classifies** S if it is a terminal object of $O_X^{(S)}$.

If we interpret this 1-categorically, then we see that indeed, a subobject classifieris precisely a morphism that classifies monics (subobjects) in the sense of this definition.

Definition 3.3. [nLa19d] A map $f : x \to y$ in an infinity category X is a **monomorphism** if the induced functor $X/f \to X/y$ is a fully faithful functor.

Lemma 3.4. *[Lur09, Example 6.1.6.2] The ∞-category $\mathcal{S}paces$ has a subobject classifier given by the monic $\Delta^0 \hookrightarrow \{0, 1\}$, where the codomain is the discrete two object Kan complex.*

A natural question that arises is whether or not ∞-toposes have subobject classifiers? It turns out that whilst this is true, it is not as relevant when working in the (∞)-categorical setting. Instead, one may want to be able to classify all maps in a higher topos. Following the discussion following [Lur09, Proposition 6.1.6.3], we see that this is often unreasonable. In a 1-category $\mathscr{C}$, due to the fact that hom sets are sets, if $Y \to X$ is not a monic, we lose information about non-trivial automorphisms of Y. In the ∞-categorical case however, as the hom spaces are now Kan complexes, this information can be retained. Another issue is that of running into size issues: by having an object classify every object in an infinity category X, [Lur09, Proposition 6.1.6.3] guarentees that every slice category $X_{/x}$ will be essentially small. To get around this, we introduce object classifiers.

Definition 3.5. [Lur09, Definition 6.1.6.4] Let X be a presentable ∞-category and κ a cardinal. A map $f : x \to y$ in X is **relatively κ-compact** if for every pullback diagram

$$\begin{array}{ccc} x' & \longrightarrow & x \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ y' & \longrightarrow & y \end{array}$$

such that Y' is κ-compact, X' is also κ-compact.

The following classification of ∞ toposes was shown by Rezk.

Theorem 3.6. *[Lur09, Theorem 6.1.6.8] Let X be an ∞-category. Then, X is an ∞-topos if and only if the following conditions hold:*

1. *X is presentable,*
2. *Colimits in X are universal,*
3. *For all sufficiently large regular cardinals κ, there exists a classifying object for the class of all relatively κ-compact morphisms in X.*

Definition 3.7. [Lur09, Definition 6.1.1.2] Let X be a presentable infinity category. We say that **colimits in X are universal** if the associated pullback functor $f : X^{/y} \to X^{/x}$ preserves small colimits, for any map that $f : x \to y$ in X.

Returning to our discussion on subobjects, we recall that a way to state the existence of a subobject classifier in a 1-category is demonstrate that the functor Sub is representable. This is the route that Rasekh takes in defining object classifiers in higher categories.

Definition 3.8. [Ras18, Definition 1.78] Let X be a higher category with finite limits and let S be a subclass of morphisms, closed under pullbacks. An object $\mathcal{U}^S$ of X is an **object classifier for** S if it represents the functor

$$((X_{/-})^S)^{\mathrm{core}} : X^{op} \to \mathcal{S}paces.$$

The category $(X/x)^S$ is full subcategory of X/x spanned by the maps in S. The functor $(-)^{core}$ takes an ∞-category and outputs the maximal subgroupoid, which we view as an object in $\mathcal{S}paces$.

This can be seen as the same notion as Lurie's object classifier in the following way: given any object $x \in X$, representability gives an equivalence

$$((X_{/x})^S)^{\mathrm{core}} \simeq X(x, \mathcal{U}^S).$$

This then gives rise to a map $\mathcal{U}^S_* \to \mathcal{U}^S$ with the property that for any map $f : x \to y$ in S, there is an essentially unique pullback square:

$$\begin{array}{ccc} x & \longrightarrow & y \\ \big\downarrow & & \big\downarrow \\ \mathcal{U}^S_* & \longrightarrow & \mathcal{U}^S \end{array}$$

This condition is precisely the condition that Lurie gives: the category $O_X^{(S)}$ has a terminal object.

Object classifiers are often called universes and can be thought of as an internal ∞-topos [nLa19e, Remark 1.1]. From the homotopy type theoretic perspective, object classifiers correspond to types of types.

Following [Ras18, Example 1.81], we can look at the object classifiers in $\mathcal{S}paces$.

Definition 3.9. Let κ be a sufficiently large cardinal. We define the following:

1. $\mathcal{S}paces^{\kappa}$ is the higher category of spaces which are κ small.
2. $\mathcal{U}^{\kappa}$ is the category $(\mathcal{S}paces^{\kappa})^{\text{core}}$.
3. $\mathcal{U}^{\kappa}_{*}$ is the category $(\mathcal{S}paces^{\kappa}_{*})^{\text{core}}$, where $\mathcal{S}paces^{\kappa}_{*}$ are κ small pointed spaces.

Lemma 3.10. *[Ras18, Example 1.81] The category $\mathcal{S}paces^{\kappa}$ has an object classifier.*

Proof. The forgetful map $\mathcal{S}paces^{\kappa}_{*} \to \mathcal{S}paces^{\kappa}$ which forgets the pointedness structure induces a map $p : \mathcal{U}^{\kappa}_{*} \to \mathcal{U}^{\kappa}$. We can then show that the functor

$$((\mathcal{S}paces_{/-})^{\kappa})^{\text{core}}$$

is representable. To do this, we first observe the following chain of equivalences:

$$\begin{aligned} ((\mathcal{S}paces_{/*})^{\kappa})^{\text{core}} &\simeq ((\mathcal{S}paces)^{\kappa})^{\text{core}}, \\ &= \mathcal{U}^{\kappa}, \\ &\simeq \mathcal{S}paces(*, \mathcal{U}^{\kappa}). \end{aligned}$$

To complete the proof, we use the property that every space is a colimit of the point and both sides commute with colimits [Ras18, Example 1.81, (2)]. $\square$

Spaces can be recovered by taking pullbacks along the map $\mathcal{U}^{\kappa}_{*} \to \mathcal{U}^{\kappa}$. We can define a κ small space X to be a map $X : \Delta^0 \to \mathcal{U}^{\kappa}$. The following pullback diagram then arises:

$$\begin{array}{ccc} \Delta^0 \times_{\mathcal{U}^{\kappa}} \mathcal{U}^{\kappa}_{*} & \longrightarrow & \mathcal{U}^{\kappa}_{*} \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow[X]{} & \mathcal{U}^{\kappa} \end{array}$$

We see that a fibre over X is then the space of all pointed spaces (X, x), with x a point of X. This space can be identified with X, meaning that the pullback is the space X.

4 Elementary ∞-toposes

So far we have only looked at higher toposes of sheaves, which we shall call Grothendieck higher toposes. As is the case with 1-topos theory, there is a weaker notion of a topos known as an elementary topos. The definition of an elementary 1-topos is motivated from a logical perspective in that it gives you enough structure to do most finitary (intuitionistic) logic. To differentiate between the two different notions, the previous ∞-topos will be refered to as **Grothendieck ∞-toposes**. In the higher case, whilst the definition of a Grothendieck higher topos is set in stone, there has been some debate around what an elementary higher topos should be, however it now seems like there is an accepted definition.

Definition 4.1. [nLa19c] An **elementary ∞-topos** is an ∞-category $\mathscr{E}$ such that the following conditions hold:

1. $\mathscr{E}$ has finite limits and colimits
2. $\mathscr{E}$ is locally cartesian closed
3. There exists a subobject classifier
4. For any morphism $f : y \to x$ in $\mathscr{E}$, there is a class of morphisms S containing f which is closed under finite limits, colimits, composition and dependent products, such that there is an object classifier for $\mathcal{U}^S$ for S.

The definition of cartesian closed is taken straight from 1-category theory - finite products and a terminal objects with the functor $x \times (-)$ having a right adjoint for every object x in $\mathscr{E}$. Locally cartesian closed here means that every slice category $\mathscr{E}/x$ is also cartesian closed. This immediately means that every elementary ∞-topos is cartesian closed via the identification of $\mathscr{E}_{/\Delta^0}$ with $\mathscr{E}$. It is discussed in an n-Category Café blog post by Shulman [Shu17] that every Grothendieck ∞-topos is an elementary ∞-topos, which means that we can prove some results in an easier fashion by showing that they hold for elementary toposes.

Lemma 4.2. *[nLa19c, Theorem 3.1] Any morphism $x \to 0$ in an elementary ∞-topos is an equivalence.*

Proof. This proof is more or less the same as the 1-categorical proof. Given a morphism $f : x \to 0$, we see that the projection $x \times 0 \to x$ has a section $\langle 1_x, f\rangle : x \to x \times 0$. Noting that the functor $x \times (-)$ is a left adjoint, it must preserve colimits [nLa19b], hence $x \times 0$ is equivalent to 0, meaning that f must be an equivalence. □

One of the key results in 1-topos theory is the Fundamental Theorem of Topos theory, which states that the slice of an elementary topos is again an elementary topos. This theorem holds in the ∞-topos setting.

Theorem 4.3. *[Ras18, Theorem 3.10] Let $\mathscr{E}$ be an elementary ∞ topos and x an object of $\mathscr{E}$. Then $\mathscr{E}/x$ is an elementary ∞-topos.*

Proof. The proof found in [Ras18, Theorem 3.10] uses a different definition of elementary ∞-topos to the one we do. However, we will give out outline of the similar properties.

The category $\mathscr{E}/x$ has finite limits and colimits, inherited from $\mathscr{E}$. We will also give an overview of the existence of a subobject classifier. If $\mathscr{E}$ has a subobject classifier given by Ω, we claim that the projection map $\pi_2 : \Omega \times x \to x$ is a subobject classifier in $\mathscr{E}/x$. Firstly, we need to determine what monics in $\mathscr{E}/x$ are: it turns out that a morphism $h : f \to g$ is a monic in $\mathscr{E}/x$ precisely when h is a monic in $\mathscr{E}$. This means if $f : y \to x$ is an object in $\mathscr{E}/x$, the restriction map $\mathrm{Sub}(f) \to \mathrm{Sub}(y)$ is going to be an equivalence of spaces (where $\mathrm{Sub}(-)$ is suitable defined). This means it suffices to show that there is an equivalence $\mathscr{E}_{/x}(f, \pi_2) \simeq \mathrm{Sub}(y)$. Rasekh then does this by exhibiting a certain adjunction which gives rise to the following chain of equivalences

$$\mathrm{Sub}(f) \xrightarrow{\simeq} \mathrm{Sub}(y) \xrightarrow{\simeq} \mathscr{E}(y, \Omega) \xrightarrow{\simeq} \mathscr{E}_{/x}(f, \pi_2)$$

This shows that $\mathrm{Sub}(f) \simeq \mathscr{E}_{/x}(f, \pi_2)$, meaning that π_2 is the subobject classifier, as required. □

5 Summary

We have defined Grothendieck ∞-toposes and looked briefly at the components in the definition. We then studied object classifiers and discussed their role and use in toposes, showing that a sub object classifer can be thought of as a special case of and object classifier. Finally, we introduced elementary ∞-toposes and looked at some results that can be proven about them, including an overview of part of the proof of the fundemental theorem of ∞-toposes. In this paper we have only scratched the surface of the rich structure of that ∞-toposes can have. Further places of research include understanding the relationship between dependent type theories and ∞-toposes, in particular looking at the internal languages of such categories.

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