HoTT Learning Seminar - 8.2 & 8.3

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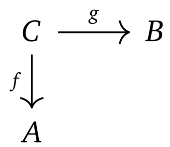
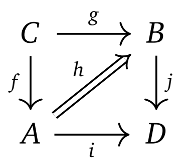
April 1, 2020

1 Recall

Definition 1.1. A **span** $\mathscr{D}$ consists of the following diagram:



Definition 1.2. A **cocone under $\mathscr{D}$ with base D** is a triple (i, j, h), represented by the following diagram:



The type of all cocones on $\mathscr{D}$ with base D is defined as:

$$\mathrm{cocone}_{\mathscr{D}}(D) :\equiv \Sigma_{i:A\to D}\Sigma_{j:B\to D}\Pi_{c:C}\, if(c) = jg(c).$$

Definition 1.3. Let $\mathscr{D}$ be a span of n-types, D be an n-type and $c : \mathrm{cocone}_{\mathscr{D}}(D)$. The pair (D, c) is a **pushout of $\mathscr{D}$ in n-types** if for every n-type E, the following map is an equivalence:

$$(D \to E) \to \mathrm{cocone}_{\mathscr{D}}(E)$$
$$t \mapsto t \circ c$$

Theorem 1.4. *[Uni13, Theorem 7.4.12] Let $\mathscr{D}$ be a span and (D, c) a pushout of $\mathscr{D}$. Then $(\|D\|_n, \|c\|_n)$ is a pushout of $\|\mathscr{D}\|_n$ in n-types*

Theorem 1.5. *The pushout of* $1 \leftarrow A \rightarrow 1$ *is the* ***suspension*** *of* A*.*

Definition 1.6. A type A is called ***n*-connected** if $||A||_n$ is contractible.

Proposition 1.7. *A type* A *is* n*-connected if and only if the map*

$$\lambda b.\lambda a.b : B \rightarrow (A \rightarrow B)$$

is an equivalence for every n*-type* B*.*

2 Connectedness of suspensions

Theorem 2.1. *If* A *is* n*-connected, then the suspension of* A *is* $(n+1)$ *connected.*

Proof. The suspension of A is given by $1 \cup^A 1$. We need to show that $||1 \cup^A 1||_{n+1}$ is contractible. Theorem [Uni13, Theorem 7.4.12] tells us that this type is a pushout of the following span in $n+1$-types:

$$\begin{array}{ccc} ||A||_{n+1} & \longrightarrow & ||1||_{n+1} \\ \downarrow & & \\ ||1||_{n+1} & & \end{array}$$

This diagram is equal to:

$$\begin{array}{ccc} ||A||_{n+1} & \longrightarrow & 1 \\ \downarrow & & \\ 1 & & \end{array}$$

Define $\mathscr{D}$ to be this span. We claim that 1 is a pushout in $(n+1)$-types of this span, which by the universal property of pushouts will give $||1 \cup^A 1||_{n+1} \simeq 1$, showing that the $(n+1)$-truncation of the suspension of A, is contractible.

By Definition 1.3, we need to show that for every $(n+1)$-type E, the following map is an equivalence:

$$\begin{aligned} (1 \rightarrow E) &\rightarrow \mathrm{cocone}_{\mathscr{D}}(E) \\ k &\mapsto (k, k, \lambda u.\mathrm{refl}_{k(*)}) \end{aligned} \tag{1}$$

We do this by finding a chain of equivalences, whose composite is equal to (1). The first map we are interested in is:

$$\begin{aligned} (1 \rightarrow E) &\rightarrow E \\ k &\mapsto k(*) \end{aligned} \tag{2}$$

We note that this is clearly an equivalence. We also have the following map, which is an equivalence (see Appendix A):

$$\begin{aligned} E &\to \Sigma_{x:E}\Sigma_{y:E} x =_E y \\ x &\to (x, x, \mathrm{refl}_x) \end{aligned} \tag{3}$$

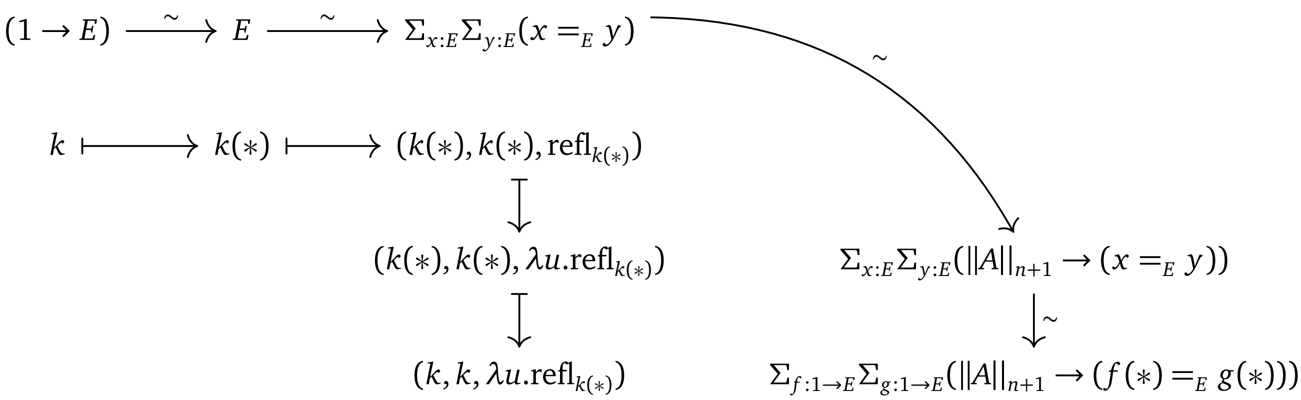
We note that $||A_{n+1}||$ is n-connected, as $||||A||_{n+1}||_n = ||A||_n = 1$. Also, $x =_E y$ is an n type as E is an $(n+1)$-type. This means that the following map is an equivalence, by Corollary [Uni13, Corollary 7.5.9]:

$$\begin{aligned} (x =_E y) &\to (||A||_{n+1} \to (x =_E y)) \\ p &\mapsto \lambda z.p \end{aligned} \tag{4}$$

In particular this means we have an equivalence:

$$\begin{aligned} \Sigma_{x:E}\Sigma_{y:E}(x =_E y) &\to \Sigma_{x:E}\Sigma_{y:E}(||A||_{n+1} \to (x =_E y)) \\ (x, y, p) &\mapsto (x, y, \lambda z.p) \end{aligned} \tag{5}$$

Putting all of this together gives the following:



We note that $\mathrm{cocone}_{\mathscr{D}}(D) = \Sigma_{f:1\to E}\Sigma_{g:1\to E}(||A||_{n+1} \to (f(*) =_E g(*)))$, and this above composite is the map (1), that we wanted to be an equivalence.

Therefore, the suspension of A is $(n+1)$-connected. □

Corollary 2.2. *For all $n : \mathbb{N}$, the n-sphere S^n is $(n-1)$-connected.*

Proof. We note that $||S^0||_{-1}$ is the propositional truncation of S^0, so there is a path between any two terms. As S^0 is non-empty, we then have $||S^0||_{-1}$ being contractible. Suppose that S^n is $(n-1)$-connected. As S^{n+1} is the suspension of S^n, Theorem 2.1 tells us that S^{n+1} is n-connected. □

3 $\pi_{k\leq n}$ of an n-connected space

Lemma 3.1. *If A is an n-type and $a : A$, then $\pi_k(A,a) = 1$ for all $k > n$.*

Proof. Recall that the loop space, $\Omega(A,a)$ of an n-type is an $n-1$ type. This means that $\Omega^k(A,a)$ is an $(n-k)$-type. As $n-k \leq -1$, so $\Omega^k(A,a)$ is an inhabited mere proposition, hence contractible. Then, $\pi_k(A,a) = \|\Omega^k(A,a)\|_0 = 1$. □

Lemma 3.2. *If A is n-connected and $a : A$, then $\pi_k(A,a) = 1$ for all $k \leq n$*

Proof. We have the following equalities:

$$\begin{aligned}
\pi_k(A,a) &= \|\Omega^k(A,a)\|_0 \\
&= \Omega^k(\|(A,a)\|_k) \\
&= \Omega^k(\|\|(A,a)\|_n\|_k) && k \leq n \implies \|-\|_k \circ \|-\|_n = \|-\|_k \\
&= \Omega^k(\|1\|_k) && A \text{ is } n\text{-connected} \\
&= \Omega^k(1) \\
&= 1.
\end{aligned}$$

□

Corollary 3.3. *$\pi_k(S^n) = 1$ for $k < n$.*

Proof. By Corollary 2.2, the n-sphere is $(n-1)$-connected. Apply Lemma 3.2. □

A Coq code

We give a coq proof that the function (3) is an equivalence.

```
Require Import HoTT.

Definition homotopy {A B : Type} (f g : A -> B) : Type :=
  forall x : A, f x = g x.

Definition isequiv {A B : Type} (f : A -> B): Type :=
  exists (g : B -> A),
  ((homotopy (g o f) (idmap)) /\ (homotopy (idmap) (f o g))).

Definition func3 (A : Type) :
  A -> exists (x : A), exists (y : A), x = y :=
  fun a : A => (a; (a; idpath)).

Goal forall (A : Type), isequiv (func3 A).
intro A. srefine (_;_).
+ intro H. apply H.
+ split.
  - intro x. reflexivity.
  - intro x. simpl. induction x.
  induction proj2_sig.
  induction proj2_sig.
  simpl. unfold func3. reflexivity.
Defined.
```

References

[Uni13] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. https://homotopytypetheory.org/book, Institute for Advanced Study, 2013.