Mathematics and Logic: A Theory of Sets

James Leslie

University of Edinburgh

s1307323@sms.ed.ac.uk

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Zermelo-Frankel Axioms

1. Extensionality: Two sets A and B are equal precisely when they have the same members.
2. Pair Set: For any two x and y, there is a set $\{x, y\}$ whose members are exactly x and y.
3. Power Set: For any set x there is a set $\mathcal{P}(x)$ of subsets of x.
4. Union: For each x there is a set $\bigcup x$ whose members are the members of members of x.
5. Subset: Suppose ϕ is a property of sets and a is some set, then there is a set $\{x \in a : \phi(x)\}$ whose members are those of a which satisfy ϕ.
6. A countably infinite set exists.
7. Replacement: Let x be any set. Let H be a well-defined operation which assigns sets to members of x. Then there is a set whose members are exactly $H(a)$ for all $a \in x$.
8. Foundation: For all non empty sets x, there exists $y \in x$ such that $y \cap x = \emptyset$.

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- $\mathbb{N}$ - ???

Peano Axioms

Constants: 0, Binary Functions: $+, \cdot$, Unitary Functions: S.

1. There is no n such that $S(n) = 0$.
2. The function S is injective.
3. For all x: $x + 0 = 0$.
4. For all x, y: $x + S(y) = S(x + y)$.
5. For all x: $x \cdot 0 = 0$.
6. For all x, y: $x \cdot S(y) = x \cdot y + x$.
7. You can do induction.

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5. $S(0) + 0 = S(0)$ — 4 ui: $x \mapsto S(0)$

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5. $S(0) + 0 = S(0)$ — 4 ui: $x \mapsto S(0)$
6. $S(0) + S(0) = S(S(0))$ — 3, 5, sub

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6. $S(0) + S(0) = S(S(0))$ — 3, 5, sub
7. Q.E.D

The End