Category of elements

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June 30, 2018

Definition 1. Given a locally small category $\mathscr{A}$ and a functor $X : \mathscr{A}^{op} \to \mathbf{Set}$, the **category of elements** of X, denoted $\mathbb{E}(X)$ or $\int^{\mathscr{A}} X$, is defined as follows:

- Objects are pairs $(A \in \mathbb{A}, x \in X(A))$,
- Morphisms $f : (A, x) \to (A', x')$ are maps $f : A \to A' \in \mathscr{A}$ such that $(Xf)(x') = x$.

Given a presheaf X, there is a projection funtor $P : \mathbb{E}(X) \to \mathscr{A}$ that sends $(A, x) \mapsto A$ and $f \mapsto f$. As a result of the property that morphisms satisfy, we can write them as $f : (A', (Xf)(x)) \to (A, x)$. It is worth noticing that if there is an $\mathbb{A}$-morphism $f : A' \to A$, then there is a unique element $x' \in X(A')$ such that there is an $\mathbb{E}(X)$-morphism $f : (A', x') \to (A, x)$, namely $x' = (Xf)(x)$.The category of elements can also be treated as a comma category.

Lemma 2. [2, Exercise 6.2.22] *There is an isomorphism* $\mathbb{E}(X) \cong (1 \Rightarrow X)$.

Proof. We look at the comma category for the following diagram:

$$\begin{array}{ccc} & & \mathbb{A}^{op} \\ & & \big\downarrow{\scriptstyle X} \\ \mathbb{1} & \xrightarrow[1]{} & \mathbf{Set} \end{array}$$

Here, $\mathbb{1}$ is the terminal category and $1 : \mathbb{1} \to \mathbf{Set}$ is the functor that selects the terminal set. This category has as objects, pairs $(A \in \mathbb{A}, x : 1 \to X(A))$ and morphisms $f : (A, x) \to (A', x')$ are commuting triangles:

$$\begin{array}{ccc} 1 & \xrightarrow{x'} & X(A') \\ & {\scriptstyle x}\searrow & \big\downarrow{\scriptstyle Xf} \\ & & X(A) \end{array}$$

That this triangle commutes is the same as stating $x = (Xf)(x')$, which is the condition above. $\square$

Proposition 3. [2, Exercise 6.2.23] *Let X be a presheaf on a locally small category. X is representable if and only if $\mathbb{E}(X)$ has a terminal object.*

Proof. The category $\mathbb{E}(X)$ has a terminal object if and only if there is an object (A, x) such that for any (A', x'), there is exactly one morphism $f : (A', x') \to (A, x)$. This is equivalent to there being an $A \in \mathscr{A}$ and $x \in X(A)$ such that for all $A' \in \mathscr{A}$, $x \in X(A')$, there is a unique morphism $f : A' \to A$ such that $(Xf)(x) = x'$. This condition is equivalent to X being representable, by [2, Corollary 4.3.2]. □

The category of elements is very useful when looking at colimits in functor categories.

Proposition 4. [2, Theorem 6.2.17] *Let $\mathbb{A}$ be small and $X : \mathbb{A}^{op} \to \mathbf{Set}$ a presheaf. Then X is the colimit of the following diagram:*

$$\mathbb{E}(X) \xrightarrow{P} \mathscr{A} \xrightarrow{H_\bullet} [\mathscr{A}^{op}, \mathbf{Set}]$$

That is, $X \cong \lim_{\to \mathbb{E}(X)}(H_\bullet \circ P)$.

We should first note that this does make sense; as $\mathscr{A}$ is small, so is $\mathbb{E}(X)$, hence a colimit does indeed exist.

Proof. We know that presheaf categories have all (small) limits and colimits, so a colimit of $H_\bullet \circ P$ exists. Let $Y \in [\mathbb{A}^{op}, \mathbf{Set}]$ be a presheaf and let $(\alpha_{(A,x)} : (H_\bullet \circ P)(A, x) \to Y)_{(A,x)\in\mathbb{E}(X)}$ be a cocone on $H_\bullet \circ P$ with vertex Y. We can simply this to have $(\alpha_{(A,x)} : (H_A \to Y)_{(A,x)\in\mathbb{E}(X)}$. This is a family of natural transformations, so for all $f : (A', x') \to (A, x)$ in $\mathbb{E}(X)$, the folowing diagram commutes

$$\begin{array}{ccc} H_{A'} & & \\ {\scriptstyle H_f}\downarrow & \searrow^{\alpha_{(A',(Xf)(x))}} & \\ H_A & \xrightarrow[\alpha_{(A,x)}]{} & Y \end{array} \tag{1}$$

By the Yoneda lemma, every natural transformation $\alpha_{(A,x)} : H_A \to Y$ corresponds to a unique element $(\alpha_{(A,x)})_A(1_A) \in Y(A)$, which we shall denote $y_{(A,x)}$. As diagram (1) commutes, it commutes for all $A \in \mathbb{A}$, so in particular it commutes for A'. This gives us the following:

$$\begin{array}{ccccc} H_{A'}(A') & & & 1_{A'} \longmapsto (\alpha_{(A',(Xf)(x))})_{A'}(1_{A'}) & (= y_{(A',(Xf)(x))}) \\ {\scriptstyle H_f(A')}\downarrow & \searrow^{(\alpha_{(A',(Xf)(x))})_{A'}} & & \downarrow & \\ H_A(A') & \xrightarrow[(\alpha_{(A,x)})_{A'}]{} & Y(A') & f \longmapsto (\alpha_{(A,x)})_{A'}(f) & \end{array} \tag{2}$$

This gives us $y_{(A',(Xf)(x))} = (\alpha_{(A,x)})_{A'}(f)$. As $\alpha_{(A,x)}$ is a natural transformation, the following square commutes:

$$
\begin{array}{ccc}
H_A(A) & \xrightarrow{H_A(f)} & H_A(A') \\
{\scriptstyle (\alpha_{(A,x)})_A}\downarrow & & \downarrow{\scriptstyle (\alpha_{(A,x)})_{A'}} \\
Y(A) & \xrightarrow[Yf]{} & Y(A')
\end{array}
\qquad
\begin{array}{ccc}
1_A & \longmapsto & f \\
\big\downarrow & & \downarrow \\
 & & (\alpha_{(A,x)})_{A'}(f) \\
(\alpha_{(A,x)})_A(1_A) & \longmapsto & (Yf)((\alpha_{(A,x)})_A(1_A))
\end{array}
\tag{3}
$$

This gives us $(Yf)(y_{(A,x)}) = (\alpha_{(A,x)})_{A'}(f)$. Combining this with the above we see that a cocone on Y is a collection of elements $(y_{(A,x)})_{(A,x)\in\mathbb{E}(X)}$ such that for any $f : (A', (Xf)(x)) \to (A, x)$ in $\mathbb{E}(X)$, $(Yf)(y_{(A,x)}) = y_{(A',(Xf)(x))}$.

An equivalent way to write $y_{(A,x)}$ is $\overline{\alpha}_A(x) : X(A) \to Y(A)$ and treat it as a function.The properties above then say for any $f : (A', (Xf)(x)) \to (A, x)$ in $\mathbb{E}(X)$, $(Yf)(\overline{\alpha}_A(x)) = \overline{\alpha}_{A'}((Xf)(x))$, that is to say the following diagram commutes for all f:

$$
\begin{array}{ccc}
X(A) & \xrightarrow{Xf} & X(A') \\
{\scriptstyle \overline{\alpha}_A}\downarrow & & \downarrow{\scriptstyle \overline{\alpha}_{A'}} \\
Y(A) & \xrightarrow[Yf]{} & Y(A')
\end{array}
\tag{4}
$$

This shows that $\overline{\alpha} : X \to Y$ is a natural transformation. As all of the above is equivalent, we see that a cocone on Y is the same as a map from X into Y, hence X is the colimit of $H_\bullet \circ P$. We can write this as equivalence formally as

$$[\mathbb{E}(X), [\mathbb{A}^{op}, \mathbf{Set}]](H_\bullet \circ P, \Delta Y) \cong [\mathbb{A}^{op}, \mathbf{Set}](X, Y).$$

This is an application of the dual of [2, Equation 6.2]. □

0.1 An equivalence

Given a set S, there is an equivalence of categories $\mathbf{Set}/S \simeq \mathbf{Set}^S$, where the latter has as objects S indexed tuples of sets. Given $(A, f : A \to S) \in \mathbf{Set}/S$, we form the tuple $(f^{-1}(s))_{s\in S}$ and given a tuple $(A_s)_{s\in S}$, we form the disjoint union $\coprod_{s\in S} A_s$ along with the function $g : \coprod_{s\in S} A_s \to S$ that sends every element in each A_s to s. This equivalence can be abstracted to categories by the following theorem.

Theorem 5. [1, Proposition 1.1.7] *Let $\mathbb{A}$ be a small category and $X : \mathbb{A}^{op} \to$ **Set** a presheaf on $\mathbb{A}$. Then there is an equivalence of categories*

$$[\mathbb{A}^{op}, \mathbf{Set}]/X \simeq [\mathbb{E}(X)^{op}, \mathbf{Set}]. \tag{5}$$

Proof. There are lots of naturality conditions that need to be checked; however, we shall ignore most of them as they are quite easy to check. We first define the following functor:

$$\begin{aligned}
\widehat{\ } : [\mathbb{A}^{op}, \mathbf{Set}]/X &\to [\mathbb{E}(X)^{op}, \mathbf{Set}] \\
(F, \alpha : F \to X) &\mapsto (\widehat{(F,\alpha)} : \mathbb{E}(X)^{op} \to \mathbf{Set}), \\
(\lambda : (F,\alpha) \to (G,\beta)) &\mapsto (\hat{\lambda} : \widehat{(F,\alpha)} \to \widehat{(G,\beta)}).
\end{aligned}$$

The functor $\widehat{(F,\alpha)}$ is defined as follows:

$$\begin{aligned}
\widehat{(F,\alpha)} : \mathbb{E}(X)^{op} &\to \mathbf{Set} \\
(A, x) &\mapsto \alpha_A^{-1}(x), \\
f : (A', (Xf)(x)) \to (A, x) &\mapsto \widehat{(F,\alpha)}(f) : \alpha_A^{-1}(x) \to \alpha_{A'}^{-1}((Xf)(x)).
\end{aligned}$$

Where $\widehat{(F,\alpha)}(f)(y) = (Ff)(y)$. The natural transformation $\hat{\lambda}$ has components $\hat{\lambda}_{(A,x)} : \alpha_A^{-1} \to \beta_A^{-1}(x)$ with $\hat{\lambda}_{(A,x)}(y) = \lambda_A(y)$. We now define a map in the other direction:

$$\begin{aligned}
\widetilde{\ } : [\mathbb{E}(X)^{op}, \mathbf{Set}] &\to [\mathbb{A}^{op}, \mathbf{Set}]/X \\
P : \mathbb{E}(X)^{op} \to \mathbf{Set} &\mapsto \left(\tilde{P}_A : \coprod_{x \in X(A)} P_x(A) \to X(A) \right)_{A \in \mathbb{A}}, \\
\lambda : P \to Q &\mapsto \left(\tilde{\lambda}_A : \coprod_{x \in X(A)} P_x(A) \to \coprod_{x \in X(A)} Q_x(A) \right)_{A \in \mathbb{A}}.
\end{aligned}$$

The functor $P_x : \mathbb{A}^{op} \to \mathbf{Set}$ is defined as $P_x(A) = P(A, x)$. This can then be made into a functor $\coprod_{x \in X(-)} P_x : \mathbb{A}^{op} \to \mathbf{Set}$. The natural transformation $\tilde{P}$ has components defined by the universal property of the coproduct. If $y \in P_x(A)$ then $\tilde{P}_A(y) = x$. The natural transformation $\tilde{\lambda}$ has components with the following action on $y \in P(A, x)$ - $\tilde{\lambda}_A(y) = \lambda_{(A,x)}(y)$.

We need to show that the composites of these functors are naturally isomorphic to the identity functors. Given $(F, \alpha) \in [\mathbb{A}^{op}, \mathbf{Set}]/X$, $\widetilde{\widehat{(F,\alpha)}}$ is a pair $(\coprod_{x \in X(-)} \alpha_{(-)}^{-1}(x), \tilde{\hat{\alpha}})$. For any $A \in \mathbb{A}$, there is a map $\varphi_A^{(F,\alpha)}$ such that the following commutes:

$$\begin{array}{ccc}
\coprod_{x \in X(A)} \alpha_A^{-1}(x) & \xrightarrow{\tilde{\hat{\alpha}}_A} & X(A) \\
\varphi_A^{(F,\alpha)} \downarrow & \nearrow_{\alpha_A} & \\
F(A) & &
\end{array}
\qquad
\begin{array}{ccc}
y & \longmapsto & \alpha_A(y) \\
\downarrow & \nearrow & \\
y & &
\end{array}$$

This makes sense as for every $x \in X(A)$, $\alpha_A^{-1}(x) \subseteq F(A)$, so $\varphi_A^{(F,\alpha)}(y)$ is an inclusion map. We notice that each $\alpha_A^{-1}(x)$ is disjoint, so $\varphi_A^{(F,\alpha)}$ is injective. It must also be surjective as for any $y \in F(A)$, $y \in \alpha_A^{-1}(\alpha_A(y))$, hence $\varphi_A^{(F,\alpha)}$ is an isomorphism for all A. We need first show that it is natural in $A \in \mathbb{A}$, so let $f : A' \to A$ be an $\mathbb{A}$-morphism. We see that the following commutes:

$$\begin{array}{ccc} \widetilde{\widehat{(F,\alpha)}}(A) & \xrightarrow{\widetilde{\widehat{(F,\alpha)}}(f)} & \widetilde{\widehat{(F,\alpha)}}(A') \\ \varphi_A^{(F,\alpha)}\big\downarrow & & \big\downarrow\varphi_{A'}^{(F,\alpha)} \\ F(A) & \xrightarrow[F(f)]{} & F(A') \end{array} \qquad \begin{array}{ccc} \coprod_{x\in X(A)} \alpha_A^{-1}(x) & \xrightarrow{\widetilde{\widehat{(F,\alpha)}}(f)} & \coprod_{x\in X(A')} \alpha_{A'}^{-1}(x) \\ \varphi_A^{(F,\alpha)}\big\downarrow & & \big\downarrow\varphi_{A'}^{(F,\alpha)} \\ F(A) & \xrightarrow[F(f)]{} & F(A') \end{array}$$

$$\begin{array}{ccc} y & \longmapsto & (Ff)(y) \\ \big\downarrow & & \big\downarrow \\ y & \longmapsto & (Ff)(y) \end{array}$$

Now we have a morphism $\varphi^{(F,\alpha)} : \widetilde{\widehat{(F,\alpha)}} \to (F,\alpha)$, for every $(F,\alpha) \in [\mathbb{A}^{op},\mathbf{Set}]$. We need to show that it is natural. Let $\lambda : (F,\alpha) \to (G,\beta)$ be a $[\mathbb{A}^{op},\mathbf{Set}]/X$-morphism. It suffices to show that the following diagram commutes for every $A \in \mathbb{A}$, which we see it does.

$$\begin{array}{ccc} \widetilde{\widehat{(F,\alpha)}}(A) & \xrightarrow{\widetilde{\widehat{\lambda}}_A} & \widetilde{\widehat{(G,\beta)}}(A) \\ \varphi_A^{(F,\alpha)}\big\downarrow & & \big\downarrow\varphi_A^{(G,\beta)} \\ F(A) & \xrightarrow[\lambda_A]{} & G(A) \end{array} \qquad \begin{array}{ccc} \coprod_{x\in X(A)} \alpha_A^{-1}(x) & \xrightarrow{\widetilde{\widehat{\lambda}}_A} & \coprod_{x\in X(A)} \beta_A^{-1}(x) \\ \varphi_A^{(F,\alpha)}\big\downarrow & & \big\downarrow\varphi_A^{(G,\beta)} \\ F(A) & \xrightarrow[\lambda_A]{} & F(A) \end{array}$$

$$\begin{array}{ccc} y \in \alpha_A^{-1}(x) & \longmapsto & \lambda_A(y) \\ \big\downarrow & & \big\downarrow \\ y & \longmapsto & \lambda_A(y) \end{array}$$

This shows that φ is natural, hence $\widetilde{\widehat{\cdot}} \cong \mathbb{1}_{[\mathbb{A}^{op},\mathbf{Set}]/X}$, naturally. The other isomorphism is shown similarly.

$\square$

References

[1] T. Leinster, *Higher operads, higher categories*, 2003.

[2] ———, *Basic category theory.* Cambridge Studies in Advanced Mathematics, Vol. 143, Cambridge University Press, 2014, 2016. Accessed: 25/06/18.